

Weighted approximation by double singular integral operators with radially defined kernels

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Abstract In this study, we present some results on the weighted pointwise convergence of a family of singular integral operators with radial kernels given in the following form:

$$L_{\lambda}(f; x, y) = \iint_{\mathbb{R}^2} f(t, s) H_{\lambda}(t - x, s - y) ds dt, \quad (x, y) \in \mathbb{R}^2, \quad \lambda \in \Lambda,$$

where Λ is a set of non-negative numbers with accumulation point λ_0 , and the function f is measurable on \mathbb{R}^2 in the sense of Lebesgue.

Keywords Generalized Lebesgue point · Radial kernel · Pointwise convergence · Rate of convergence

Mathematics Subject Classification Primary: 41A35 · Secondary: 41A25 · 45P05 · 47A58 · 47B65

Introduction

The approximation of functions by integral operators with positive definite kernels is widely used in many branches of mathematics, such as approximation theory, representation

theory, theory of differential equations, Fourier analysis, and singular integral theory. Besides, it is well known that Fourier analysis is used and has many applications in medicine and engineering; more specifically, magnetic resonance imaging (MRI) and fingerprint identification are the familiar examples in those areas, respectively. Particularly, the great importance of singular integral theory, which originated in Fourier analysis, must be emphasized here. In the construction stage of Fourier series of the functions, the following integral is obtained at the end of consecutive operations, that is

$$L_{\lambda}(f; x) = \int_{-\pi}^{\pi} f(t) K_{\lambda}(t, x) dt, \quad x \in [-\pi, \pi], \quad \lambda \in \mathbb{N}, \quad (1)$$

where $K_{\lambda}(t)$ denotes a kernel satisfying some conditions similar to usual approximate identities. Singular integrals consist of the different settings of the operators of type (1) with an appropriate singularity assumption on the kernel. For mentioned applications and related other applications concerning the usage of approximation theory in natural and applied sciences, the authors refer to [1–11].

The pointwise approximation problem may be seen as a problem of representing functions at some characteristic points, such as point of continuity, Lebesgue point, generalized Lebesgue point, and μ -generalized Lebesgue point. μ -generalized Lebesgue point, among others, comes to the fore. Actually, depending on the choice of the function $\mu(t)$, definitions of the remaining points can be easily obtained. In practice, there are two major investigation methods related to the pointwise convergence of integral-type operators, such as operators of type (1).

The first method can be described as fixing the variable x within the operator of type (1). In other words, we pick a point in the domain of integration and it represents all other

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points of the same kind. Therefore, the convergence of the operator is investigated almost everywhere in the domain of integration. This method is used in many works, such as Rydzewska [12], Mamedov [13], Butzer and Nessel [14], and Uysal et al. [15].

The second method also known as Fatou-type convergence can be described as restricting the pointwise convergence to some subsets of the plane [16]. Therefore, a sensitive convergence analysis is obtained. For some studies related to Fatou-type convergence, the authors refer to [16–18].

Now, we summarize some of the works in which this method is harnessed.

In [19], Taberski, who indicated the importance of singular integrals in Fourier series in his works, investigated the pointwise approximation of periodic and integrable functions on $\langle -\pi, \pi \rangle$, where $\langle -\pi, \pi \rangle$ is an arbitrary closed, semi-closed, or open interval. The work used a two parameter family of singular integral operators of the form:

$$L_\lambda(f; x) = \int_{-\pi}^{\pi} f(t) K_\lambda(t - x) dt, \quad x \in \langle -\pi, \pi \rangle, \quad \lambda \in \Lambda, \quad (2)$$

where $K_\lambda : \mathbb{R} \rightarrow \mathbb{R}_0^+$ denotes a family of periodic kernels satisfying suitable conditions, and Λ is a given set of non-negative numbers with accumulation point λ_0 .

Taberski [20], which gave enthusiasm to researchers, advanced his analysis to double singular integral operators of the form:

$$L_\lambda(f; x, y) = \iint_Q f(t, s) H_\lambda(t - x, s - y) ds dt, \quad (x, y) \in Q, \quad (3)$$

where $Q = \langle -\pi, \pi \rangle \times \langle -\pi, \pi \rangle$ is an arbitrary closed, semi-closed, or open region, $H_\lambda : \mathbb{R}^2 \rightarrow \mathbb{R}_0^+$ stands for a family of kernels, and $\lambda \in \Lambda$ is a set of non-negative numbers with accumulation point λ_0 . Indicated paper also contains two-dimensional generalization of well-known Natanson's lemma. Then, Siudut [21, 22] presented considerable theorems by the aid of these results. Note that Rydzewska [23] also improved her previous work [12] using the results of [20], and she obtained the rate of convergence of the operators of type (3). Later on, Taberski [24] obtained the weighted pointwise approximation of some integral operators using a weight function satisfying some conditions. Moreover, this study was seen as a continuation and two-dimensional analogue of [25]. In recent papers [26–28], the kernel functions within the operators of type (3) were defined as radial functions and the domains of integration were replaced by an arbitrary region $\langle a, b \rangle \times \langle c, d \rangle$. As

concerns the study of integral operators in several settings, the reader may also see [29–35].

This study is a continuation and further generalization of [26]. Besides, the current manuscript deals with Fatou-type pointwise convergence of a family of singular integral operators with radial kernels given in the following form:

$$L_\lambda(f; x, y) = \iint_{\mathbb{R}^2} f(t, s) H_\lambda(t - x, s - y) ds dt, \quad (x, y) \in \mathbb{R}^2, \quad \lambda \in \Lambda, \quad (4)$$

where $H_\lambda(t - x, s - y) = K_\lambda(\sqrt{(t - x)^2 + (s - y)^2})$, and Λ is a set of non-negative numbers with accumulation point λ_0 . Here, $f \in L_{1,\varphi}(\mathbb{R}^2)$ and $L_{1,\varphi}(\mathbb{R}^2)$ are the space of all measurable functions for which $\left| \frac{f}{\varphi} \right|$ is integrable provided $\varphi : \mathbb{R}^2 \rightarrow \mathbb{R}^+$ is a weight function which is bounded on any bounded subset of \mathbb{R}^2 .

The paper is organized as follows: In Sect. 2, we introduce the fundamental definitions. In Sect. 3, we prove the pointwise convergence of $L_\lambda(f; x, y)$ to $f(x_0, y_0)$. In Sect. 4, we establish the rate of convergence of the operators of type (4).

Preliminaries

In this section, we introduce the main definitions used in this paper.

Definition 1 A function $H \in L_1(\mathbb{R}^2)$ is said to be radial if there exists a function $K : \mathbb{R}_0^+ \rightarrow \mathbb{R}$, such that $H(t, s) = K(\sqrt{t^2 + s^2})$ almost everywhere [36].

Now, we give another characterization of μ -generalized Lebesgue point using the μ -generalized Lebesgue point definition given in [23].

Definition 2 Let $\delta_0 > 0$ be an arbitrary fixed real number. A μ -generalized Lebesgue point of a locally integrable function $g : \mathbb{R}^2 \rightarrow \mathbb{R}$ is a point $(x_0, y_0) \in \mathbb{R}^2$ satisfying

$$\lim_{(h,k) \rightarrow (0,0)} \frac{1}{\mu_1(h)\mu_2(k)} \int_0^h \int_0^k |g(t + x_0, s + y_0) - g(x_0, y_0)| ds dt = 0,$$

where $\mu_1(h) = \int_0^h \rho_1(t) dt > 0$, $0 < h < \delta_0$ and $\rho_1(t)$ is an integrable and non-negative function on $[0, \delta_0]$, and similarly, $\mu_2(k) = \int_0^k \rho_2(s) ds > 0$, $0 < k < \delta_0$ and $\rho_2(s)$ is an integrable and non-negative function on $[0, \delta_0]$.

Example 1 Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ is given as follows:

$$f(t, s) = \begin{cases} \frac{1}{\sqrt{|t||s|}}, & \text{if } ts \neq 0, \\ 1, & \text{if } ts = 0. \end{cases}$$



and $\varphi : \mathbb{R}^2 \rightarrow \mathbb{R}^+$ is given by $\varphi(t, s) = (1 + |t|)(1 + |s|)$. Therefore, we have

$$\frac{f(t, s)}{\varphi(t, s)} = \begin{cases} \frac{1}{(1 + |t|)(1 + |s|)}, & \text{if } ts = 0, \\ \frac{1}{\sqrt{|t|}(1 + |t|)\sqrt{|s|}(1 + |s|)}, & \text{if } ts \neq 0. \end{cases}$$

Using the definition of μ -generalized Lebesgue point and taking $\rho_1(t) = \left\{t^{\frac{1}{2}}e^t\right\}'_t$ and $\rho_2(s) = \left\{s^{\frac{1}{2}}e^s\right\}'_s$, we see that origin is a μ -generalized Lebesgue point of $f \in L_{1,\varphi}(\mathbb{R}^2)$. On the other hand, one can check that origin is not a generalized Lebesgue point for any choice of $\alpha \in [0, 1)$. Therefore, this example shows that the nature of μ -generalized Lebesgue point depends on $\rho_1(t)$ and $\rho_2(s)$. For the analysis of one-dimensional counterpart of the function $\frac{f}{\varphi}$, we refer the reader to see [37].

Definition 3 (Class A_φ) Let $H_\lambda : \mathbb{R}^2 \rightarrow \mathbb{R}_0^+$ be a radial function, i.e., there exists a function $K_\lambda : \mathbb{R}_0^+ \rightarrow \mathbb{R}_0^+$, such that $H_\lambda(t, s) := K_\lambda(\sqrt{t^2 + s^2})$ holds for almost everywhere on \mathbb{R}^2 for each fixed $\lambda \in \Lambda$. Furthermore, let $H_\lambda : \mathbb{R}^2 \rightarrow \mathbb{R}_0^+$ be a family of radial kernels, which are integrable on \mathbb{R}^2 and the weight function $\varphi : \mathbb{R}^2 \rightarrow \mathbb{R}^+$, which is bounded on arbitrary bounded subsets of \mathbb{R}^2 , satisfies the following inequality:

$$\varphi(u + t, v + s) \leq \varphi(u, v)\varphi(t, s), (u, v) \in \mathbb{R}^2, (t, s) \in \mathbb{R}^2, \quad (5)$$

and there hold:

- a. For any given $(x_0, y_0) \in \mathbb{R}^2$

$$\lim_{(x,y,\lambda) \rightarrow (x_0,y_0,\lambda_0)} \frac{1}{\varphi(x_0, y_0)} \iint_{\mathbb{R}^2} \varphi(t, s) K_\lambda \left(\sqrt{(t-x)^2 + (s-y)^2} \right) ds dt = 1.$$

- b. $\forall \xi > 0$,

$$\lim_{\lambda \rightarrow \lambda_0} \sup_{\xi \leq \sqrt{t^2 + s^2}} \left[\varphi(t, s) K_\lambda \left(\sqrt{t^2 + s^2} \right) \right] = 0.$$

- c. $\forall \xi > 0$,

$$\lim_{\lambda \rightarrow \lambda_0} \left[\iint_{\xi \leq \sqrt{t^2 + s^2}} \varphi(t, s) K_\lambda \left(\sqrt{t^2 + s^2} \right) ds dt \right] = 0.$$

- d. $H_\lambda(t, s)$ is monotonically increasing with respect to s on $(-\infty, 0]$, and similarly, $H_\lambda(t, s)$ is monotonically increasing with respect to t on $(-\infty, 0]$ for any $\lambda \in \Lambda$.

Analogously, $H_\lambda(t, s)$ is bimonotonically increasing with respect to (t, s) on $[0, \infty) \times [0, \infty)$ and $(-\infty, 0] \times (-\infty, 0]$ and bimonotonically decreasing with respect to (t, s) on $[0, \infty) \times (-\infty, 0]$ and $(-\infty, 0] \times [0, \infty)$ for any $\lambda \in \Lambda$.

- e. $\|\varphi K_\lambda\|_{L_1(\mathbb{R}^2)} \leq M < \infty$, for all $\lambda \in \Lambda$.
f. For fixed $(t_0, s_0) \in \mathbb{R}^2$, $H_\lambda(t_0, s_0)$ tends to infinity, as λ tends to λ_0 .

Note that throughout this paper, we suppose that the function $H_\lambda(t, s)$ belongs to class A_φ .

Remark 1 If the function $g : \mathbb{R}^2 \rightarrow \mathbb{R}$ is bimonotonically increasing on $[\alpha_1, \alpha_2] \times [\beta_1, \beta_2] \subset \mathbb{R}^2$, then the following equality

$$V(g; [\alpha_1, \alpha_2] \times [\beta_1, \beta_2]) = \bigvee_{\alpha_1}^{\alpha_2} \bigwedge_{\beta_1}^{\beta_2} (g(t, s)) = g(\alpha_1, \beta_1) - g(\alpha_1, \beta_2) - g(\alpha_2, \beta_1) + g(\alpha_2, \beta_2)$$

holds. On the other hand, if the function $g : \mathbb{R}^2 \rightarrow \mathbb{R}$ is bimonotonically decreasing on $[\alpha_1, \alpha_2] \times [\beta_1, \beta_2] \subset \mathbb{R}^2$, then the following equality

$$V(g; [\alpha_1, \alpha_2] \times [\beta_1, \beta_2]) = \bigvee_{\alpha_1}^{\alpha_2} \bigwedge_{\beta_1}^{\beta_2} (g(t, s)) = g(\alpha_1, \beta_2) - g(\alpha_1, \beta_1) - g(\alpha_2, \beta_2) + g(\alpha_2, \beta_1)$$

holds [20, 38].

Convergence at characteristic points

The following lemma gives the existence of the operators defined by (4).

Lemma 1 If $f \in L_{1,\varphi}(\mathbb{R}^2)$, then the operator $L_\lambda(f; x, y)$ defines a continuous transformation acting on $L_{1,\varphi}(\mathbb{R}^2)$.

Proof Since $L_\lambda(f; x, y)$ is linear, it is sufficient to show that the expression given by

$$\|L_\lambda\|_\varphi = \sup_{f \neq 0} \frac{\|L_\lambda(f; x, y)\|_{L_{1,\varphi}(\mathbb{R}^2)}}{\|f\|_{L_{1,\varphi}(\mathbb{R}^2)}}$$

is bounded.

The following expression

$$\|f\|_{L_{1,\varphi}(\mathbb{R}^2)} = \iint_{\mathbb{R}^2} \left| \frac{f(t, s)}{\varphi(t, s)} \right| ds dt, \quad (6)$$

defines a norm in the space $L_{1,\varphi}(\mathbb{R}^2)$ [24]. Using inequality (5) and Fubini's theorem [14], we have



$$\begin{aligned}
\|L_\lambda(f; x, y)\|_{L_{1,\varphi}(\mathbb{R}^2)} &= \iint_{\mathbb{R}^2} \frac{1}{\varphi(x, y)} \left| \iint_{\mathbb{R}^2} f(t, s) H_\lambda(t - x, s - y) ds dt \right| dy dx \\
&= \iint_{\mathbb{R}^2} \frac{1}{\varphi(x, y)} \left| \iint_{\mathbb{R}^2} f(t, s) K_\lambda \left(\sqrt{(t-x)^2 + (s-y)^2} \right) ds dt \right| dy dx \\
&\leq \iint_{\mathbb{R}^2} \frac{1}{\varphi(x, y)} \left(\iint_{\mathbb{R}^2} |f(t+x, s+y)| K_\lambda \left(\sqrt{t^2 + s^2} \right) ds dt \right) dy dx \\
&= \iint_{\mathbb{R}^2} K_\lambda \left(\sqrt{t^2 + s^2} \right) \left(\iint_{\mathbb{R}^2} \left| \frac{f(t+x, s+y)}{\varphi(x, y)} \right| dy dx \right) ds dt \\
&= \iint_{\mathbb{R}^2} K_\lambda \left(\sqrt{t^2 + s^2} \right) \left(\iint_{\mathbb{R}^2} \left| \frac{f(t+x, s+y)}{\varphi(x, y)} \right| \frac{\varphi(t+x, s+y)}{\varphi(t+x, s+y)} dy dx \right) ds dt \\
&\leq \iint_{\mathbb{R}^2} K_\lambda \left(\sqrt{t^2 + s^2} \right) \left(\iint_{\mathbb{R}^2} \left| \frac{f(t+x, s+y)}{\varphi(t+x, s+y)} \right| \frac{\varphi(x, y)\varphi(t, s)}{\varphi(x, y)} dy dx \right) ds dt \\
&= \iint_{\mathbb{R}^2} K_\lambda \left(\sqrt{t^2 + s^2} \right) \varphi(t, s) ds dt \iint_{\mathbb{R}^2} \left| \frac{f(t+x, s+y)}{\varphi(t+x, s+y)} \right| dy dx \\
&\leq M \|f\|_{L_{1,\varphi}(\mathbb{R}^2)}.
\end{aligned}$$

Thus, the proof is completed. \square

The following theorem gives a Fatou-type pointwise convergence of the integral operators of type (4) at μ -generalized Lebesgue point of $f \in L_{1,\varphi}(\mathbb{R}^2)$.

Theorem 1 *If (x_0, y_0) is a μ -generalized Lebesgue point of $f \in L_{1,\varphi}(\mathbb{R}^2)$, then*

$$\lim_{(x,y,\lambda) \rightarrow (x_0,y_0,\lambda_0)} L_\lambda(f; x, y) = f(x_0, y_0)$$

on any set Z on which the function

$$\begin{aligned}
&\int_{x_0-\delta y_0-\delta}^{x_0+\delta y_0+\delta} \int_{x_0-\delta}^{x_0+\delta} K_\lambda \left(\sqrt{(t-x)^2 + (s-y)^2} \right) \rho_1(|x_0-t|) \rho_2(|y_0-s|) ds dt \\
&+ 2\mu_2(|y_0-y|) \int_{x_0-\delta}^{x_0+\delta} K_\lambda(|t-x|) \rho_1(|x_0-t|) dt \\
&+ 2\mu_1(|x_0-x|) \int_{y_0-\delta}^{y_0+\delta} K_\lambda(|s-y|) \rho_2(|y_0-s|) ds \\
&+ 4K_\lambda(0)\mu_1(|x_0-x|)\mu_2(|y_0-y|)
\end{aligned}$$

is bounded as (x, y, λ) tends to (x_0, y_0, λ_0) .

Proof Let $|x_0 - x| < \frac{\delta}{2}$ and $|y_0 - y| < \frac{\delta}{2}$, for any $0 < \delta < \delta_0$. Furthermore, let $0 < x_0 - x < \frac{\delta}{2}$ and $0 < y_0 - y < \frac{\delta}{2}$ for any $0 < \delta < \delta_0$. Since $(x_0, y_0) \in \mathbb{R}^2$ is a μ -generalized Lebesgue point of function $f \in L_{1,\varphi}(\mathbb{R}^2)$, for all given $\varepsilon > 0$, there exists $\delta > 0$, such that for all h and k satisfying $0 < h, k \leq \delta$, we have the following inequality:

$$\int_{x_0}^{x_0+h} \int_{y_0-k}^{y_0} \left| \frac{f(t, s)}{\varphi(t, s)} - \frac{f(x_0, y_0)}{\varphi(x_0, y_0)} \right| ds dt < \varepsilon \mu_1(h) \mu_2(k). \quad (7)$$

Write

$$\begin{aligned}
&|L_\lambda(f; x, y) - f(x_0, y_0)| \\
&= \left| \iint_{\mathbb{R}^2} f(t, s) H_\lambda(t - x, s - y) ds dt - f(x_0, y_0) \right|.
\end{aligned}$$

Adding and subtracting the expression given by $\frac{f(x_0, y_0)}{\varphi(x_0, y_0)} \iint_{\mathbb{R}^2} \varphi(t, s) H_\lambda(t - x, s - y) ds dt$ to the right-hand side of the above equality, we have

$$\begin{aligned}
&|L_\lambda(f; x, y) - f(x_0, y_0)| \\
&= \left| \iint_{\mathbb{R}^2} f(t, s) \frac{\varphi(t, s)}{\varphi(t, s)} H_\lambda(t - x, s - y) ds dt - f(x_0, y_0) \frac{\varphi(x_0, y_0)}{\varphi(x_0, y_0)} \right. \\
&\quad \left. + \frac{f(x_0, y_0)}{\varphi(x_0, y_0)} \iint_{\mathbb{R}^2} \varphi(t, s) H_\lambda(t - x, s - y) ds dt \right. \\
&\quad \left. - \frac{f(x_0, y_0)}{\varphi(x_0, y_0)} \iint_{\mathbb{R}^2} \varphi(t, s) H_\lambda(t - x, s - y) ds dt \right| \\
&\leq \iint_{\mathbb{R}^2} \left| \frac{f(t, s)}{\varphi(t, s)} - \frac{f(x_0, y_0)}{\varphi(x_0, y_0)} \right| \varphi(t, s) H_\lambda(t - x, s - y) ds dt \\
&\quad + \left| \frac{f(x_0, y_0)}{\varphi(x_0, y_0)} \right| \left| \iint_{\mathbb{R}^2} \varphi(t, s) H_\lambda(t - x, s - y) ds dt - \varphi(x_0, y_0) \right| \\
&= I_1 + I_2.
\end{aligned}$$

Since H_λ is a radial function, we may write

$$\begin{aligned}
I_2 &= \left| \frac{f(x_0, y_0)}{\varphi(x_0, y_0)} \right| \left| \iint_{\mathbb{R}^2} \varphi(t, s) H_\lambda(t - x, s - y) ds dt - \varphi(x_0, y_0) \right| \\
&= \left| \frac{f(x_0, y_0)}{\varphi(x_0, y_0)} \right| \left| \iint_{\mathbb{R}^2} \varphi(t, s) K_\lambda \left(\sqrt{(t-x)^2 + (s-y)^2} \right) ds dt - \varphi(x_0, y_0) \right|.
\end{aligned}$$

In view of condition (a) of class A_φ , $I_2 \rightarrow 0$ as (x, y, λ) tends to (x_0, y_0, λ_0) . The integral I_1 can be written in the form:

$$\begin{aligned}
I_1 &= \left\{ \iint_{\mathbb{R}^2 \setminus B_\delta} + \iint_{B_\delta} \right\} \left| \frac{f(t, s)}{\varphi(t, s)} - \frac{f(x_0, y_0)}{\varphi(x_0, y_0)} \right| \varphi(t, s) H_\lambda(t - x, s - y) ds dt \\
&= I_{11} + I_{12},
\end{aligned}$$

where

$$B_\delta = \left\{ (t, s) : (t - x_0)^2 + (s - y_0)^2 < \delta^2, (x_0, y_0) \in \mathbb{R}^2 \right\}.$$

In view of definition of H_λ , and using inequality (5), we have

$$\begin{aligned} I_{11} &= \iint_{\mathbb{R}^2 \setminus B_\delta} \left| \frac{f(t, s)}{\varphi(t, s)} - \frac{f(x_0, y_0)}{\varphi(x_0, y_0)} \right| \varphi(t, s) H_\lambda(t - x, s - y) ds dt \\ &\leq \varphi(x, y) \iint_{\mathbb{R}^2 \setminus B_\delta} \left| \frac{f(t, s)}{\varphi(t, s)} - \frac{f(x_0, y_0)}{\varphi(x_0, y_0)} \right| \varphi(t - x, s - y) \\ &\quad \times K_\lambda \left(\sqrt{(t - x)^2 + (s - y)^2} \right) ds dt. \end{aligned}$$

Now, using the initial assumptions given as $0 < |x_0 - x| < \frac{\delta}{2}$ and $0 < |y_0 - y| < \frac{\delta}{2}$, we may define the following set:

$$A_\delta = \left\{ (x, y) : (x - x_0)^2 + (y - y_0)^2 < \frac{\delta^2}{2}, (x_0, y_0) \in \mathbb{R}^2 \right\}.$$

Taking into account the geometric representations of the sets B_δ and A_δ gives the inclusion relation $\mathbb{R}^2 \setminus B_\delta \subseteq \mathbb{R}^2 \setminus C_\delta$, where

$$C_\delta = \left\{ (t, s) : (t - x)^2 + (s - y)^2 < \frac{\delta^2}{2}, (x, y) \in A_\delta \right\}.$$

Therefore, we have the following inequality:

$$\begin{aligned} I_{11} &\leq \varphi(x, y) \iint_{\mathbb{R}^2 \setminus C_\delta} \left| \frac{f(t, s)}{\varphi(t, s)} - \frac{f(x_0, y_0)}{\varphi(x_0, y_0)} \right| \varphi(t - x, s - y) \\ &\quad \times K_\lambda \left(\sqrt{(t - x)^2 + (s - y)^2} \right) ds dt \\ &= \varphi(x, y) \iint_{\frac{\delta}{\sqrt{2}} \leq \sqrt{u^2 + v^2}} \left| \frac{f(u + x, v + y)}{\varphi(u + x, v + y)} - \frac{f(x_0, y_0)}{\varphi(x_0, y_0)} \right| \varphi(u, v) \\ &\quad \times K_\lambda \left(\sqrt{u^2 + v^2} \right) dv du \\ &\leq \varphi(x, y) \sup_{\frac{\delta}{\sqrt{2}} \leq \sqrt{u^2 + v^2}} \left[\varphi(u, v) K_\lambda \left(\sqrt{u^2 + v^2} \right) \right] \|f\|_{L_{1, \varphi}(\mathbb{R}^2)} \\ &\quad + \varphi(x, y) \left| \frac{f(x_0, y_0)}{\varphi(x_0, y_0)} \right| \iint_{\frac{\delta}{\sqrt{2}} \leq \sqrt{u^2 + v^2}} \varphi(u, v) K_\lambda \left(\sqrt{u^2 + v^2} \right) dv du. \end{aligned}$$

Consequently, by conditions (b) and (c) of class A_φ , and using boundedness of φ , $I_{11} \rightarrow 0$ as $(x, y, \lambda) \rightarrow (x_0, y_0, \lambda_0)$.

Now, we prove that I_{12} tends to zero, as (x, y, λ) tends to (x_0, y_0, λ_0) . Since $\varphi(t, s)$ is bounded on B_δ , it is easy to see that the following inequality

$$I_{12} \leq \sup_{(t, s) \in B_\delta} \varphi(t, s) \iint_{B_\delta} \left| \frac{f(t, s)}{\varphi(t, s)} - \frac{f(x_0, y_0)}{\varphi(x_0, y_0)} \right| H_\lambda(t - x, s - y) ds dt$$

holds for I_{12} . Thus, we have

$$\begin{aligned} I_{12} &\leq \sup_{(t, s) \in B_\delta} \varphi(t, s) \left\{ \int_{x_0}^{x_0 + \delta} \int_{y_0 - \delta}^{y_0} + \int_{x_0 - \delta}^{x_0} \int_{y_0}^{y_0 + \delta} \right\} \\ &\quad \times \left| \frac{f(t, s)}{\varphi(t, s)} - \frac{f(x_0, y_0)}{\varphi(x_0, y_0)} \right| H_\lambda(t - x, s - y) ds dt \\ &\quad + \sup_{(t, s) \in B_\delta} \varphi(t, s) \left\{ \int_{x_0 - \delta}^{x_0} \int_{y_0}^{y_0 + \delta} + \int_{x_0}^{x_0 + \delta} \int_{y_0}^{y_0 + \delta} \right\} \\ &\quad \times \left| \frac{f(t, s)}{\varphi(t, s)} - \frac{f(x_0, y_0)}{\varphi(x_0, y_0)} \right| H_\lambda(t - x, s - y) ds dt \\ &= \sup_{(t, s) \in B_\delta} \varphi(t, s) (I_{121} + I_{122} + I_{123} + I_{124}). \end{aligned}$$

Let us consider the integral I_{121} .

Let us define the function $V(t, s)$ by

$$V(t, s) := \int_{x_0}^t \int_s^{y_0} \left| \frac{f(u, v)}{\varphi(u, v)} - \frac{f(x_0, y_0)}{\varphi(x_0, y_0)} \right| dv du.$$

In view of inequality (7), the following expression

$$|V(t, s)| \leq \varepsilon \mu_1(t - x_0) \mu_2(y_0 - s), \quad (8)$$

where $0 < t - x_0 \leq \delta$ and $0 < y_0 - s \leq \delta$, holds. From Theorem 2.6 in [20], we can write

$$\begin{aligned} I_{121} &= (L) \int_{x_0}^{x_0 + \delta} \int_{y_0 - \delta}^{y_0} \left| \frac{f(t, s)}{\varphi(t, s)} - \frac{f(x_0, y_0)}{\varphi(x_0, y_0)} \right| H_\lambda(t - x, s - y) ds dt \\ &= (LS) \int_{x_0}^{x_0 + \delta} \int_{y_0 - \delta}^{y_0} H_\lambda(t - x, s - y) d[-V(t, s)], \end{aligned}$$

where LS denotes Lebesgue–Stieltjes integral. Applying integration by parts (see Theorem 2.2, p. 100 in [20]) to the Lebesgue–Stieltjes integral, we have

$$\begin{aligned} |I_{121}| &= \left| \int_{x_0}^{x_0 + \delta} \int_{y_0 - \delta}^{y_0} H_\lambda(t - x, s - y) d[-V(t, s)] \right| \\ &\leq \int_{x_0}^{x_0 + \delta} \int_{y_0 - \delta}^{y_0} |V(t, s)| |dH_\lambda(t - x, s - y)| \\ &\quad + \int_{x_0}^{x_0 + \delta} |V(t, y_0 - \delta)| |dH_\lambda(t - x, y_0 - \delta - y)| \\ &\quad + \int_{y_0 - \delta}^{y_0} |V(x_0 + \delta, s)| |dH_\lambda(x_0 - x + \delta, s - y)| \\ &\quad + |V(x_0 + \delta, y_0 - \delta)| H_\lambda(x_0 + \delta - x, y_0 - \delta - y). \end{aligned}$$



If we apply inequality (8) to the last inequality and make change of variables, then we have

$$\begin{aligned}
 |I_{121}| \leq & \varepsilon \int_{x_0-x}^{x_0+\delta-x} \int_{y_0-\delta-y}^{y_0-y} \mu_1(t+x-x_0) \mu_2(y_0-s-y) |dH_\lambda(t, s)| \\
 & + \varepsilon \mu_2(\delta) \int_{x_0-x}^{x_0+\delta-x} \mu_1(t+x-x_0) |dH_\lambda(t, y_0-\delta-y)| \\
 & + \varepsilon \mu_1(\delta) \int_{y_0-\delta-y}^{y_0-y} \mu_2(y_0-s-y) |dH_\lambda(x_0+\delta-x, s)| \\
 & + \varepsilon \mu_1(\delta) \mu_2(\delta) H_\lambda(x_0+\delta-x, y_0-\delta-y).
 \end{aligned}$$

Let us define the following variations:

$$\begin{aligned}
 P_1(t, s) &:= \begin{cases} \bigvee_t^{x_0+\delta-x} \bigvee_{y_0-\delta-y}^s (H_\lambda(u, v)), & x_0-x \leq t < x_0+\delta-x \\ & y_0-\delta-y < s \leq y_0-y \\ 0, & \text{otherwise.} \end{cases} \\
 P_2(t) &:= \begin{cases} \bigvee_t^{x_0+\delta-x} (H_\lambda(u, y_0-\delta-y)), & x_0-x \leq t < x_0+\delta-x \\ 0, & \text{otherwise.} \end{cases} \\
 P_3(s) &:= \begin{cases} \bigvee_{y_0-\delta-y}^s (H_\lambda(x_0-x+\delta, v)), & y_0-\delta-y < s \leq y_0-y \\ 0, & \text{otherwise.} \end{cases}
 \end{aligned}$$

Taking above variations into account and applying bivariate integration by parts method to the last inequality, we have

$$\begin{aligned}
 |I_{121}| \leq & -\varepsilon \int_{x_0-x}^{x_0+\delta-x} \int_{y_0-\delta-y}^{y_0-y} [P_1(t, s) + P_2(t) + P_3(s) \\
 & + H_\lambda(x_0-x+\delta, y_0-\delta-y)] \\
 & \times \{\mu_1(t-x_0+x)\}'_t \{\mu_2(y_0-s-y)\}'_s ds dt \\
 & = \varepsilon(i_1 + i_2 + i_3 + i_4).
 \end{aligned}$$

Using Remark 1 and condition (d) of class A_φ , we get

$$\begin{aligned}
 i_1 + i_2 + i_3 + i_4 = & - \int_{x_0-x}^{x_0+\delta-x} \int_{y_0-\delta-y}^0 K_\lambda(\sqrt{t^2+s^2}) \\
 & \{\mu_1(t+x-x_0)\}'_t \{\mu_2(y_0-s-y)\}'_s ds dt \\
 & + \int_{x_0-x}^{x_0+\delta-x} \int_0^{y_0-y} (K_\lambda(\sqrt{t^2+s^2}) - 2K_\lambda(|t|)) \\
 & \{\mu_1(t+x-x_0)\}'_t \times \{\mu_2(y_0-s-y)\}'_s ds dt.
 \end{aligned}$$

Hence, the following inequality holds for I_{121} (for the similar situation, see [20, 23]):

$$\begin{aligned}
 |I_{121}| \leq & \varepsilon \int_{x_0}^{x_0+\delta} \int_{y_0-\delta}^{y_0} K_\lambda(\sqrt{(t-x)^2 + (s-y)^2}) \\
 & \times \rho_1(t-x_0) \rho_2(|y_0-s|) ds dt \\
 & + 2\varepsilon \mu_2(|y_0-y|) \int_{x_0}^{x_0+\delta} K_\lambda(|t-x|) \rho_1(t-x_0) dt.
 \end{aligned}$$

Analogous computations for I_{122} , I_{123} , and I_{124} give us

$$\begin{aligned}
 |I_{122}| \leq & \varepsilon \int_{x_0-\delta}^{x_0} \int_{y_0-\delta}^{y_0} K_\lambda(\sqrt{(t-x)^2 + (s-y)^2}) \rho_1(x_0-t) \rho_2(y_0-s) ds dt \\
 & + 2\varepsilon \mu_2(|y_0-y|) \int_{x_0-\delta}^{x_0} K_\lambda(|t-x|) \rho_1(x_0-t) dt \\
 & + 2\varepsilon \mu_1(|x_0-x|) \int_{y_0-\delta}^{y_0} K_\lambda(|s-y|) \rho_2(y_0-s) ds \\
 & + 4\varepsilon K_\lambda(0) \mu_1(|x_0-x|) \mu_2(|y_0-y|),
 \end{aligned}$$

$$\begin{aligned}
 |I_{123}| \leq & \varepsilon \int_{x_0-\delta}^{x_0} \int_{y_0}^{y_0+\delta} K_\lambda(\sqrt{(t-x)^2 + (s-y)^2}) \\
 & \times \rho_1(x_0-t) \rho_2(s-y_0) ds dt \\
 & + 2\varepsilon \mu_1(|x_0-x|) \int_{y_0}^{y_0+\delta} K_\lambda(|s-y|) \rho_2(s-y_0) ds,
 \end{aligned}$$

$$\begin{aligned}
 |I_{124}| \leq & \varepsilon \int_{x_0}^{x_0+\delta} \int_{y_0}^{y_0+\delta} K_\lambda(\sqrt{(t-x)^2 + (s-y)^2}) \\
 & \times \rho_1(t-x_0) \rho_2(s-y_0) ds dt.
 \end{aligned}$$

Hence, the following inequality is obtained for I_{12} :

$$\begin{aligned}
 |I_{12}| \leq & \varepsilon \sup_{(t,s) \in B_\delta} \varphi(t, s) \left\{ \int_{x_0-\delta}^{x_0+\delta} \int_{y_0-\delta}^{y_0+\delta} K_\lambda(\sqrt{(t-x)^2 + (s-y)^2}) \right. \\
 & \times \rho_1(|x_0-t|) \rho_2(|y_0-s|) ds dt \\
 & + 2\mu_2(|y_0-y|) \int_{x_0-\delta}^{x_0+\delta} K_\lambda(|t-x|) \rho_1(|x_0-t|) dt \\
 & + 2\mu_1(|x_0-x|) \int_{y_0-\delta}^{y_0+\delta} K_\lambda(|s-y|) \rho_2(|y_0-s|) ds \\
 & \left. + 4K_\lambda(0) \mu_1(|x_0-x|) \mu_2(|y_0-y|) \right\}.
 \end{aligned}$$

In addition, the last inequality is obtained for other cases of the assumptions $|x_0-x| < \frac{\delta}{2}$ and $|y_0-y| < \frac{\delta}{2}$. The

remaining part of the proof is obvious by the hypotheses. Thus, the proof is completed. \square

Rate of convergence

In this section, we give a theorem concerning the rate of pointwise convergence.

Theorem 2 Assume that the hypotheses of Theorem 1 are satisfied. Let

$$\begin{aligned} \Delta(\lambda, \delta, x, y) = & \int_{x_0-\delta}^{x_0+\delta} \int_{y_0-\delta}^{y_0+\delta} K_\lambda \left(\sqrt{(t-x)^2 + (s-y)^2} \right) \\ & \times \rho_1(|x_0-t|) \rho_2(|y_0-s|) ds dt \\ & + 2\mu_2(|y_0-y|) \int_{x_0-\delta}^{x_0+\delta} K_\lambda(|t-x|) \rho_1(|x_0-t|) dt \\ & + 2\mu_1(|x_0-x|) \int_{y_0-\delta}^{y_0+\delta} K_\lambda(|s-y|) \rho_2(|y_0-s|) ds \\ & + 4K_\lambda(0)\mu_1(|x_0-x|)\mu_2(|y_0-y|) \end{aligned}$$

for $0 < \delta < \delta_0$, and the following conditions are satisfied:

1. $\Delta(\lambda, \delta, x, y) \rightarrow 0$ as $(x, y, \lambda) \rightarrow (x_0, y_0, \lambda_0)$ for some $\delta > 0$.
2. Letting $(x, y, \lambda) \rightarrow (x_0, y_0, \lambda_0)$, we have

$$\left| \frac{1}{\varphi(x_0, y_0)} \iint_{\mathbb{R}^2} \varphi(t, s) K_\lambda \left(\sqrt{(t-x)^2 + (s-y)^2} \right) ds dt - 1 \right| = o(\Delta(\lambda, \delta, x, y)).$$

3. For every $\xi > 0$

$$\sup_{\xi \leq \sqrt{t^2+s^2}} \left[\varphi(t, s) K_\lambda \left(\sqrt{t^2+s^2} \right) \right] = o(\Delta(\lambda, \delta, x, y))$$

as $(x, y, \lambda) \rightarrow (x_0, y_0, \lambda_0)$.

4. For every $\xi > 0$

$$\iint_{\xi \leq \sqrt{t^2+s^2}} \varphi(t, s) K_\lambda \left(\sqrt{t^2+s^2} \right) ds dt = o(\Delta(\lambda, \delta, x, y))$$

as $(x, y, \lambda) \rightarrow (x_0, y_0, \lambda_0)$.

Then, at each μ -generalized Lebesgue point of $f \in L_{1,\varphi}(\mathbb{R}^2)$, we have

$$|L_\lambda(f; x, y) - f(x_0, y_0)| = o(\Delta(\lambda, \delta, x, y)).$$

as $(x, y, \lambda) \rightarrow (x_0, y_0, \lambda_0)$.

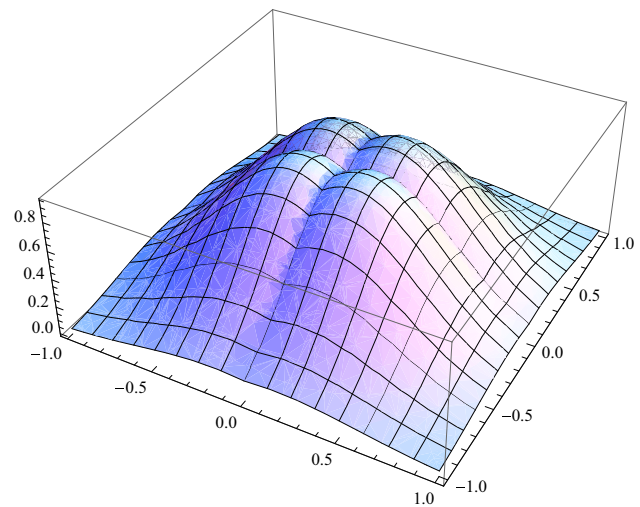


Fig. 1 Illustration of condition (3)

Proof The result is obvious by the hypotheses of Theorem 1. \square

Example 2 Let $\Lambda = (0, \infty)$, $\lambda_0 = 0$, the weight function $\varphi : \mathbb{R}^2 \rightarrow \mathbb{R}^+$ is given by $\varphi(t, s) = (1 + |t|)(1 + |s|)$ and the kernel function $H_\lambda : \mathbb{R}^2 \rightarrow \mathbb{R}_0^+$ is given by $H_\lambda(t, s) = \frac{1}{4\pi\lambda} e^{-\frac{(t^2+s^2)}{4\lambda}}$. To verify that $H_\lambda(t, s)$ satisfies the hypotheses of Theorem 1, see [21]. Let $(x_0, y_0) = (0, 0)$, $\rho_1(t) = 1$ and $\rho_2(s) = 1$. Therefore, $\mu_1(t) = t$ and $\mu_2(s) = s$. First, we give the graphical illustrations of the conditions (3) and (4). The assumptions in Fig. 1 are as follows: $\lambda = 0.1$ and $-1 \leq t, s \leq 1$. Therefore, $\varphi(t, s)H_{0.1}(t, s) = \frac{1}{0.4\pi} e^{-\frac{(t^2+s^2)}{0.4}} (1 + |t|)(1 + |s|)$.

The assumptions in Fig. 2 are as follows: $\lambda = 0.1$ and $-10 \leq t, s \leq 10$.

The Fig. 1 and Fig. 2 are generated by using the software Wolfram Mathematica 7. Note that

$$\lim_{\lambda \rightarrow 0} \iint_{\mathbb{R}^2} \frac{1}{4\lambda\pi} e^{-\frac{(t^2+s^2)}{4\lambda}} (1 + |t|)(1 + |s|) ds dt = 1.$$

Now, we focus on condition (1). Computing $\Delta(\lambda, \delta, x, y)$ gives

$$\begin{aligned} \Delta(\lambda, \delta, x, y) = & \frac{1}{\pi\lambda} |x||y| + \frac{1}{2\sqrt{\lambda\pi}} \left(\operatorname{Erf} \left(\frac{\delta-y}{2\sqrt{\lambda}} \right) + \operatorname{Erf} \left(\frac{\delta+y}{2\sqrt{\lambda}} \right) \right) \\ = & \frac{1}{4} \left(\operatorname{Erf} \left(\frac{\delta-x}{2\sqrt{\lambda}} \right) + \operatorname{Erf} \left(\frac{\delta+x}{2\sqrt{\lambda}} \right) \right) \\ & \times \left(\operatorname{Erf} \left(\frac{\delta-y}{2\sqrt{\lambda}} \right) + \operatorname{Erf} \left(\frac{\delta+y}{2\sqrt{\lambda}} \right) \right) \\ & + \frac{1}{2\sqrt{\lambda\pi}} \left(\operatorname{Erf} \left(\frac{\delta-x}{2\sqrt{\lambda}} \right) + \operatorname{Erf} \left(\frac{\delta+x}{2\sqrt{\lambda}} \right) \right). \end{aligned}$$



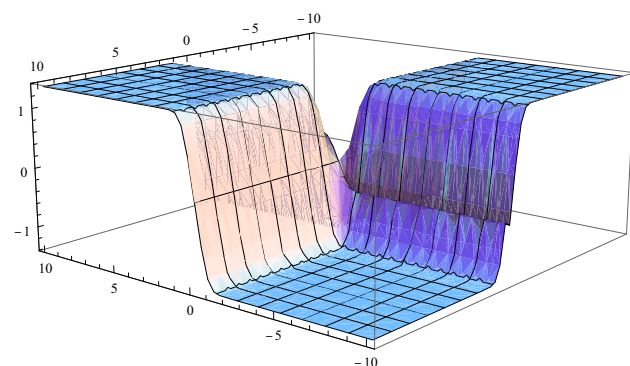


Fig. 2 Illustration of condition (4)

Here, the function $\text{Erf}(x)$ is given by

$$\text{Erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt.$$

To find the numbers $\delta > 0$, we let $\Delta(\lambda, \delta, x, y) \rightarrow 0$ as $(x, y, \lambda) \rightarrow (0, 0, 0)$. Therefore, if $\delta = o(\sqrt{\lambda})$ and $|x| = |y| = o(\sqrt{\lambda})$, then we obtain

$$\lim_{(x,y,\lambda) \rightarrow (0,0,0)} \Delta(\lambda, \delta, x, y) = 0.$$

Hence, $\Delta(\lambda, \delta, x, y) = O(\lambda^c)$, $c > \frac{1}{2}$. Using these results and evaluating the supremum value of the function $\varphi(t, s)K_\lambda(\sqrt{t^2 + s^2})$ in terms of λ , we have

$$\begin{aligned} & \lim_{\lambda \rightarrow 0} \left[\sup_{\frac{\delta}{\sqrt{2}} \leq \sqrt{t^2 + s^2}} \left[\varphi(t, s)K_\lambda(\sqrt{t^2 + s^2}) \right] \right] \\ &= \left(1 + \frac{1}{2}(-1 + \sqrt{1 + 8\lambda}) \right) e^{-\frac{(-1 + \sqrt{1 + 8\lambda})^2}{16\lambda}} \\ &= 0. \end{aligned}$$

Therefore, we obtain the desired result for the condition (3), that is

$$3. \quad \sup_{\xi \leq \sqrt{t^2 + s^2}} \varphi(t, s)K_\lambda(\sqrt{t^2 + s^2}) = o(\lambda^c), c > \frac{1}{2} \text{ as } (x, y, \lambda) \rightarrow (x_0, y_0, \lambda_0).$$

The conditions (2) and (4) are verified with the same method. Finally, we obtain

$$|L_\lambda(f; x, y) - f(x_0, y_0)| = o(\lambda^c), c > \frac{1}{2}.$$

Concluding remark

In this paper, we investigated the weighted pointwise convergence and the rate of convergence for the family of double singular integral operators of the form (4). This

study may be seen as a continuation and generalization of the previous studies, such as [26]. For this aim, we used two-dimensional counterparts of some concepts given in one-dimensional case, such as monotonicity and integration by parts method. Next, since the approximation results and the character of the kernel function are related, a special class of kernel functions, called class A_φ , has been defined. Therefore, the main result is presented as Theorem 1. Using this theorem, we obtained the rate of pointwise convergence and gave an example including graphical illustration.

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Compliance with ethical standards

Conflict of interest The authors declare that they have no competing interests.

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